

Lecture 28 Highlights Phys 402

Helium-4 is a Boson. It has two protons and two neutrons in the nucleus, as well as two electrons orbiting the nucleus. All of these particles are spin-1/2. However they pair up into either spin singlet ($S=0$) or spin triplet ($S=1$) states, as discussed in section 5.2 of Griffiths). Thus it has integer spin. Helium-4 is also a quantum fluid – it will not solidify at any temperature at a pressure of 1 atmosphere. This can be understood qualitatively in terms of the position-momentum uncertainty relation $\sigma_x \sigma_{p_x} \geq \hbar/2$, and the small mass of the He atom. Forming a solid requires that each atom has a small value for σ_x . This in turn enhances σ_{p_x} , and because of the small mass of the atom, the resulting velocity is large, allowing the atom to escape its lattice site. ^4He has many unusual macroscopic thermodynamic properties that are governed by quantum mechanics.

Some of these properties include (check out the [videos](#) on the class web site),

- 1) The absence of boiling below the “lambda transition” at $T_\lambda = 2.2$ K,
- 2) Flow with zero viscosity through “superleaks”,
- 3) Finite viscosity when measured by a rotating disk suspended by a torsional oscillator in the fluid,
- 4) Superfluid film creep,
- 5) Thermo-mechanical effects, including the fountain effect.

Why is Helium-4 a superfluid? We consider a large collection of ^4He atoms in the liquid state inside a box of dimensions $a \times a \times a$. This is a ‘gas’ of many identical Bosons with overlapping wavefunctions.

The liquid is held in a cube of sides $a \times a \times a = V$. We must now find the energies and degeneracies of the states of the system. Assume that the Helium atoms act like free particles in a box (3-dimensional infinite square well). Assume that ‘weak interactions’ occur between the atoms to enable the system to be ergodic (i.e. to explore all microscopic configurations consistent with the fixed energy and number constraints).

The single particle states are given by solutions to the Schrodinger equation with Hamiltonian $H = -\frac{\hbar^2}{2m} \nabla^2 + V$, where $V = \begin{cases} 0 & \text{for points inside the box} \\ \infty & \text{for points on the wall of the box} \end{cases}$. This

problem has been solved several times before in this class. It is solved by standard separation of variables techniques, and using the fact that the wavefunction goes to zero at the walls. Here we will use the standing wave solution to the TISE, as opposed to the running wave solution. The resulting single-particle wavefunctions are of the form $\psi(x, y, z) = A \sin\left(\frac{n_x \pi x}{a}\right) \sin\left(\frac{n_y \pi y}{a}\right) \sin\left(\frac{n_z \pi z}{a}\right)$, where n_x, n_y, n_z are positive integers.

The energy of the state is given by a triplet of integers as $E = \frac{\hbar^2 k^2}{2m}$, with $k^2 = k_x^2 + k_y^2 + k_z^2$, and $k_x = \frac{n_x \pi}{a}$, $k_y = \frac{n_y \pi}{a}$, $k_z = \frac{n_z \pi}{a}$, with $n_x = 1, 2, 3, \dots$, $n_y = 1, 2, 3, \dots$, $n_z = 1, 2, 3, \dots$. The states are labeled by points on a regular cubic lattice of spacing π/a in k-space. Each state takes up a volume of $\left(\frac{\pi}{a}\right) \times \left(\frac{\pi}{a}\right) \times \left(\frac{\pi}{a}\right) = \left(\frac{\pi}{a}\right)^3 = \frac{\pi^3}{V}$ in k-space.

All states with the same energy (i.e. degenerate) lie on a spherical octant of radius k in momentum space. We can calculate how many states are within an octant in k -space of radius k as follows. The number of states up to k is:

$$G(k) = \frac{\text{Volume of octant in } k\text{-space}}{\text{Volume occupied per state in } k\text{-space}} = \frac{\frac{1}{8} \frac{4}{3} \pi k^3}{(\pi/a)^3}. \quad \text{Written in terms of energy}$$

$E = \frac{\hbar^2 k^2}{2m}$, this becomes $G(E) = \frac{V}{6\pi^2} \left(\frac{2mE}{\hbar^2} \right)^{3/2}$. The degeneracy is the rate at which new states are added as the sphere expands. This density of states (another name for degeneracy) is given by

$$g(E) = \frac{dG}{dE} = 2\pi V \left(\frac{2m}{\hbar^2} \right)^{3/2} E^{1/2}.$$

The density of states times a differential of energy dE , i.e. $g(E)dE$, tells us how many states lie between energy E and $E + dE$. Note that this density of states is the same as that for the free electron Fermi gas, except for a factor of 2 in that case accounting for the spin degeneracy, which we don't have here.

For a box of many identical Fermions with overlapping wavefunctions the Pauli exclusion principle dictates that each Fermion occupies a unique single-particle state with a distinct list of quantum numbers. For identical Bosons the situation is very different. There is no constraint on how many particles can occupy the same state. In this case we have to use a result from finite-temperature statistical mechanics to figure out how many Bosons will be in each state. We label the states by the letter 's', which stands for the list of quantum numbers (n_x, n_y, n_z) (we shall assume that the spin of the ^4He atom is zero). We know the most likely occupation numbers n_s (how many atoms are in state s) for the states of such a system:

$$n_s = \frac{g_s}{e^{(E_s - \mu)/k_B T} - 1}$$

From the above analysis, we know the states of the system "s", the energy levels E_s , and the degeneracies of those levels g_s . There are two new quantities that we have not encountered before in our study of quantum mechanics, namely the temperature T and the chemical potential μ .

First we take a guess at the sign of the chemical potential μ . Let's assume that the lowest energy state of the system is at energy 0, i.e. $E_1 = 0$. Now suppose $\mu = 0$ as well. This leads to the following result for the most likely occupation number of the ground state:

$n_1 = \frac{g_1}{e^{(0-0)/k_B T} - 1} \rightarrow \infty$. This does not make sense, since the total number of ^4He particles in the box is fixed at N . Now suppose $\mu > 0$. In this case the most likely occupation

number for the ground state is $n_1 = \frac{g_1}{e^{-|\mu|/k_B T} - 1} < 0$. In other words, the most likely occupation number for the ground state is negative, which does not make any sense. Hence we are forced to conclude that $\mu < 0$ since only in this case do we get a non-negative and

finite occupation number for the ground state of the system. This is an important point that we will return to later.

We have already enumerated all of the states available to the system, found their energies and degeneracies. The next step is to enforce the fixed number constraint to determine the chemical potential μ (at some arbitrary temperature T).

We can now find the chemical potential μ by enforcing the number constraint for the particles in the box: $N = \sum_{s=1}^{\infty} n_s = \sum_{s=1}^{\infty} \frac{g_s}{e^{(E-\mu)/kT} - 1}$. In other words, we count up the occupation number of all the single-particle states of the system and make sure that it equals the total number of particles in the box. Rather than carry out this tedious sum, we will make a continuum approximation and convert the (discrete) sum on k (which can be used to label the states s) to an integral on (continuous) energy. Thus we get:

$$N = \int_0^{\infty} \frac{g(E)dE}{e^{(E-\mu)/kT} - 1}. \quad (1)$$

Note that we are taking the ground state of the system ($n_x = n_y = n_z = 1$) to be at zero energy. Plugging in the density of states for a particle in a 3D box yields:

$$N = 2\pi V \left(\frac{2m}{h^2} \right)^{3/2} (k_B T)^{3/2} \int_0^{\infty} \frac{x^{1/2} dx}{e^x e^{-\mu/kT} - 1},$$

where we have taken the additional step of defining the dimensionless variable $x \equiv E/kT$.

The integral can be solved by pulling out the two exponentials in the denominator:

$$I = \int_0^{\infty} \frac{x^{1/2} dx}{e^x e^{-\mu/kT} (1 - e^{-x} e^{\mu/kT})}. \text{ Note that since } \mu \text{ must be negative (see above) and}$$

x is greater than or equal to zero, the factor $e^{-x} e^{\mu/kT}$ is less than 1 in magnitude. Therefore the following expansion will converge:

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \dots \text{ where in this case } z = e^{-x} e^{\mu/kT}. \text{ This leads to an infinite}$$

number of integrals, each of which is closely related to the Gamma function, and yields the following result for the number constraint:

$$N = 2\pi V \left(\frac{2m}{h^2} \right)^{3/2} (k_B T)^{3/2} \Gamma\left(\frac{3}{2}\right) f(\mu/k_B T), \quad (2)$$

where $f(\mu/k_B T) \equiv \sum_{p=1}^{\infty} \frac{e^{p\mu/k_B T}}{p^{3/2}}$, and $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$. The function f has a maximum value

when it has zero argument, and that value is about 2.612 (see the [plot](#) on the web site).

Equation (2) allows one to find the chemical potential μ for a given number density of Bosons N/V and temperature T . However, this implicit equation for μ has problems in the limit of low temperatures. The right hand side of the equation goes to zero because the function f is bounded above and nothing else is temperature dependent. This shows that there has been a major flaw in the theory up to this point. The problem was in making the continuum approximation in Equation (1). The continuum version leaves out one very important state – the ground state at $E = 0$. This state takes up the burden of holding all

of the particles in the limit as $T \rightarrow 0$! The correct expression for the particle number becomes:

$$N = n_1(T) + 2\pi V \left(\frac{2m}{h^2} \right)^{3/2} (k_B T)^{3/2} \Gamma\left(\frac{3}{2}\right) f(\mu/k_B T)$$

$$= \frac{1}{e^{-\mu/k_B T} - 1} + 2\pi V \left(\frac{2m}{h^2} \right)^{3/2} (k_B T)^{3/2} \Gamma\left(\frac{3}{2}\right) f(\mu/k_B T) \quad (3)$$

Note from Eq. (3) that the occupation number of the ground state (n_1) can be made arbitrarily large by taking the chemical potential very close to 0 (but still negative!). We can estimate the “crisis temperature” as the point at which the function $f(\mu/k_B T)$ is forced to take on its maximum value of 2.612. This gives for the “critical temperature” T_c :

$$T_c = \left(\frac{N/V}{2\pi \left(\frac{2mk_B}{h^2} \right)^{3/2} \Gamma\left(\frac{3}{2}\right) 2.612} \right)^{2/3}$$

For ^4He , with $N/V = 2 \times 10^{28} / \text{m}^3$, this gives $T_c = 3.1\text{K}$, which is remarkably close to the experimental value for the λ -transition, $T_\lambda = 2.2\text{K}$. Similar calculations can be done for a gas of cold Na atoms, as found in atom traps. There the mass of the atom is about 23 amu and the density that can be achieved experimentally is much lower, on the order of $N/V = 10^{20} / \text{m}^3$, giving a predicted crisis temperature of $T_c = 1.5 \mu\text{K}$. The observed condensation temperature is about $2 \mu\text{K}$.

The phenomenon of macroscopic occupation of the ground state in the limit of zero temperature in a collection of many identical Bosons with overlapping wavefunctions is called Bose-Einstein Condensation (BEC).

Superfluid ^4He Properties

The unusual properties of superfluid helium can be semi-quantitatively understood in terms of the two-fluid model. At any temperature below T_λ the fluid is made up of two inter-penetrating but non-interacting fluids. The superfluid is the set of all ^4He atoms in the ground state, while the normal fluid is the set of all ^4He atoms in the excited states inside the box. Each fluid has its own density, and they are constrained to add up to the total density: $\rho_s + \rho_n = \rho$. The superfluid has no entropy, flows with zero viscosity, and has no turbulence (for slow enough flow rates). The normal fluid has viscosity and can carry entropy. An experiment that measures the moment of inertia of a set of parallel disks rotating as a torsional pendulum in the fluid shows that the densities are temperature dependent. Above T_λ there is only normal fluid. Below T_λ the superfluid density monotonically increases from zero at the expense of the normal fluid density, and eventually all the fluid is converted to superfluid in the limit of zero temperature.

The superleak is made up of small pores that pin the normal fluid because of its viscosity. The superfluid can move through the small pores because of its zero viscosity and lack of a boundary layer. The oscillating disks placed in the fluid shows an enhanced

moment of inertia because it entrains the normal fluid and drags it back and forth as it oscillates. The superfluid remains at rest as the disk oscillates.

A critical flow velocity of about 5 cm/s exists for superfluid ^4He . For flow velocities greater than this it is possible to create excitations in the superfluid because enough energy is available to begin promoting ^4He atoms out of the ground state, thus taking energy away from the flow. A finite energy gap in the excitation spectrum prevents this dissipation mechanism from acting at low flow rates.

The properties of superfluid ^4He can also be understood in terms of a macroscopic quantum wavefunction $\psi(\vec{r})$. The full many-body wavefunction for the He atoms in a box involves $\sim 10^{23}$ degrees of freedom. Under suitable conditions, this wavefunction can be approximated by that of a single quantum entity (the superfluid) with a phase-coherent wavefunction that extends over the entire size of the system. This is a wavefunction whose magnitude and phase can vary with position, and whose magnitude squared is equal to the local superfluid density: $|\psi(\vec{r})|^2 = \rho_s(\vec{r})$. The wavefunction is complex and can be written as: $\psi(\vec{r}) = \sqrt{\rho_s(\vec{r})}e^{i\theta(\vec{r})}$. In discussion we started with this wavefunction and calculated

the superfluid current density as $\vec{J}_s = \frac{\hbar}{m} \rho_s \vec{\nabla} \theta = \rho_s \vec{v}_s$, m is the mass of the ^4He atom and \vec{v}_s is the superfluid velocity field. From this we see that a mass current of the superfluid arises from a gradient in the phase of the macroscopic quantum wavefunction, rather than a pressure gradient! One can also show that the circulation of the super-flow is quantized as: $\kappa = \oint \vec{v}_s \cdot d\vec{l} = \frac{nh}{m}$, where $n = 0, \pm 1, \pm 2, \dots$. A rotating bucket of superfluid ^4He will acquire circulation in units of h/m called superfluid *vortices* (see the web site for [pictures](#)). Vortices are topological defects in the fluid flow that preserve the irrotational (or curl-free) nature of the flow: $\vec{\nabla} \times \vec{v}_s = 0$.